The Ordered Median Problem

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Motivation: Location Theory

Basic facility location problem

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- Find the „best“ locations to open p new facilities in order to satisfy the demand of all clients

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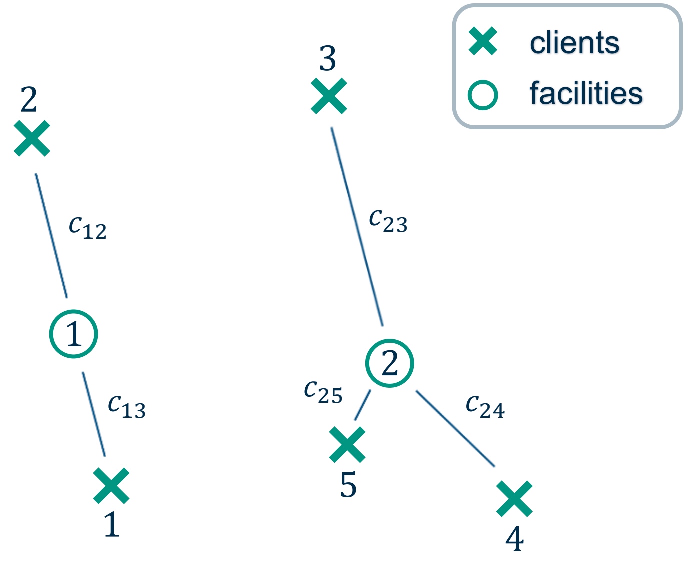
- p-median (min sum of all costs)



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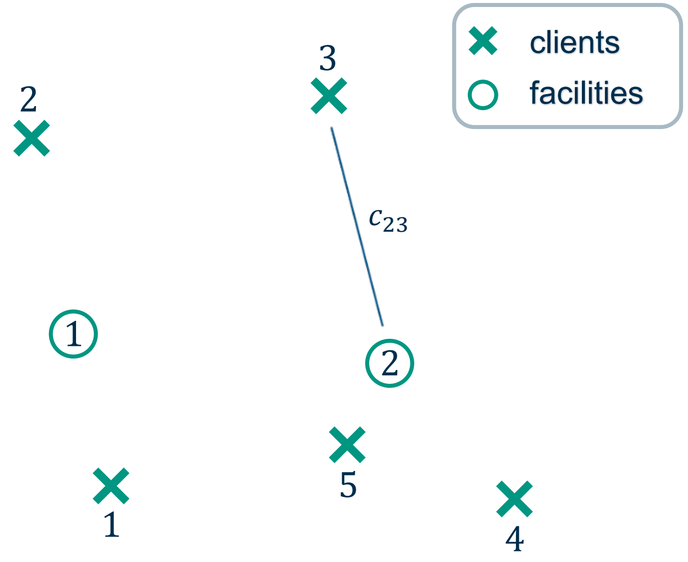

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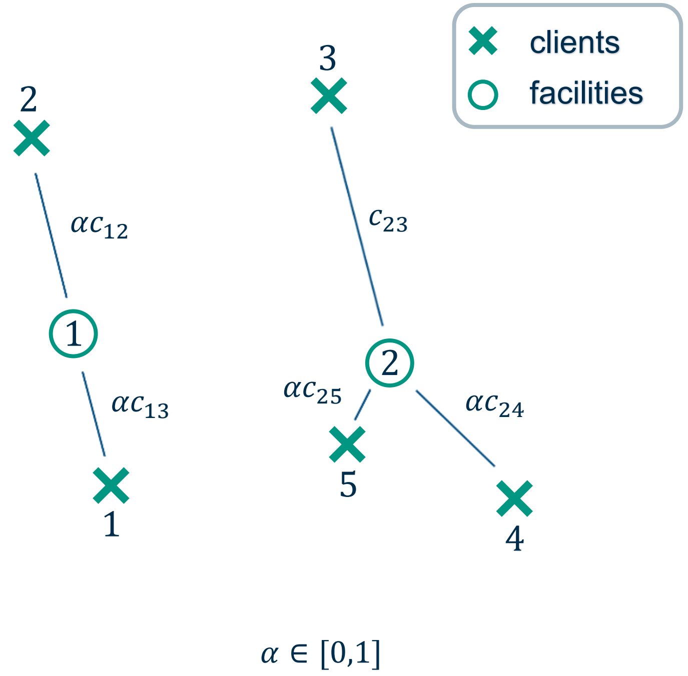

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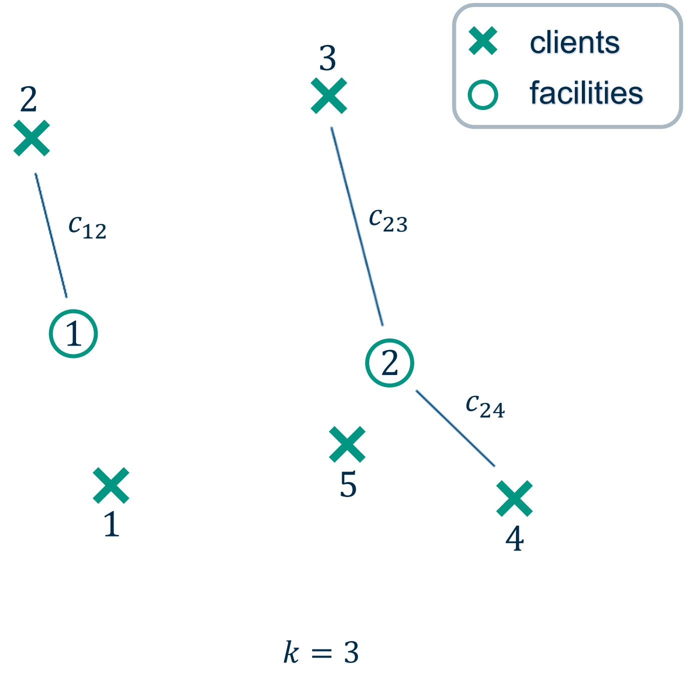
- k-centrum (min k largest costs)



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Definitions

Definition (ordered median function OMf)

For a vector $x \in \mathbb{R}^n$, let $x_{ord} = (x_{(1)}, x_{(2)}, \dots, x_{(n)})$ denote the ordered version of x with $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$.

Let $\lambda \in \mathbb{R}^n$ be an objective vector. The following function is defined to be an OMf.

$$f_\lambda : \mathbb{R}^n \to \mathbb{R}, \qquad f_\lambda(x) = \langle \lambda, x_{ord} \rangle = \lambda_1 x_{(1)} + \lambda_2 x_{(2)} + \cdots + \lambda_n x_{(n)}$$

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Examples

- $\lambda = (1, 1, \dots, 1)$ → p-median
- $\lambda = (0, \dots, 0, 1)$ → p-center
- $\lambda = (\alpha, \dots, \alpha, 1)$ → α-cent-dian
- $\lambda = (0, \dots, 0, \underbrace{1, \dots, 1}_{k \text{ times}})$ → k-centrum

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Basic Properties of OMfs

Let $f_\lambda, f_\mu : \mathbb{R}^n \to \mathbb{R}$ both be an OMf. Then:

- f_λ is non-linear
- f_λ is piecewise linear (in part., linear on $K_\pi \coloneqq \{x \in \mathbb{R}^n \mid x_{\pi(1)} \leq \cdots \leq x_{\pi(n)}\}$ for every permutation $\pi \in S_n$)
- f_λ is continuous on $\mathbb{R}^n$
- f_λ is symmetric in the sense that for any $x \in \mathbb{R}^n$: $f_\lambda(x) = f_\lambda(x_{ord})$
- f_λ is convex $\Leftrightarrow \lambda_1 \leq \cdots \leq \lambda_n$
- for $c_1, c_2 \in \mathbb{R}$, the function $c_1 f_\lambda + c_2 f_\mu$ is an OMf
- if $\{f_{\lambda^r}\}_r$ is a sequence of OMfs that pointwise converges to a function f, then f is an OMf
- f_λ is a DCH function (difference of two convex positively homogeneous functions)

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