Appendix A Additional Definitions, Lemmas, and Proofs for Section 3

A.1 Unit-sloped paths and Lemma A.1

Definition A.1 ((Recurring) unit-sloped path).
A unit-sloped path of length $2i$ is a path in $\mathbb{R}^2$ from $(0,0)$ to $(2i, s_{2i})$ consisting only of line segments between $(k-1, s_{k-1})$ and (k, s_k) for $k = 1, 2, \ldots, 2i$ where $s_k = s_{k-1} + 1$ or $s_k = s_{k-1} - 1$ and $s_0 = 0$. A recurring unit-sloped path of length $2i$ is a unit-sloped path of length $2i$ that ends in $(2i, 0)$, i.e., it has $s_{2i} = 0$. △

Note that if we restrict $s_k \geq 0$ for all $k = 0, 1, 2, \ldots, 2i$, this definition coincides with that of the well-known Dyck path (see, e.g., Deutsch (1999), Deutsch and Shapiro (2001)). Figure 4 shows an example for a recurring unit-sloped path and a Dyck path, respectively.

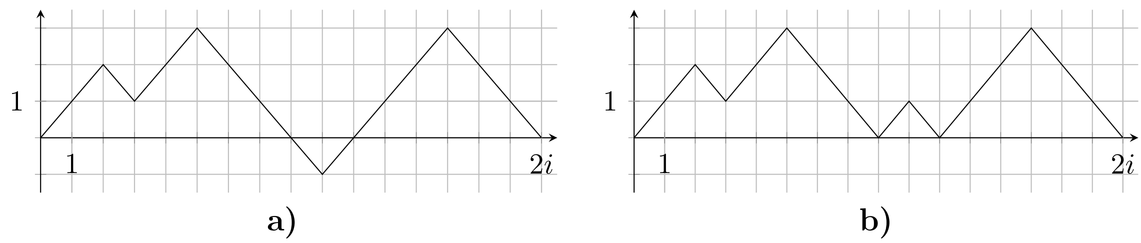

Figure 1: Recurring unit sloped paths. **a)** General path. and **b)** Dyck path.

Lemma A.1.

a) The number of recurring unit-sloped paths of length $2i$ which have $s_k \geq 0$ for all $k = 0, 1, 2, \ldots, 2i$ is C_i.

b) The number of recurring unit-sloped paths of length $2i$ which have $s_k \geq -1$ for all $k = 0, 1, 2, \ldots, 2i$ is C_{i+1}.

Proof.

a) See Michaels and Rosen (1991), chapter 7, pages 115 and 116.

b) The proof is a straightforward consequence of the fact that the number of recurring unit-sloped paths of length $2i$ with $s_k \geq -1$ for $k = 0, 1, 2, \ldots, 2i$ is the number of all recurring unit-sloped paths of length $2i$ minus the number of all recurring unit-sloped paths of length $2i$ which hit the number -2 at least once. As a result of the reflection principle, counting the paths from $(0,0)$ to $(2i, 0)$ hitting -2 at least once is the same as counting the paths from $(0,0)$ to $(2i, -4)$ hitting -2 at least once. But any such path must hit -2 at some point, i.e., we are computing the total number of paths from $(0,0)$ to $(2k, -4)$. In total, we obtain that the number of recurring unit-sloped paths of length $2i$ with $s_k \geq -1$ for $k = 0, 1, 2, \ldots, 2i$ is equal to the total number of paths from $(0,0)$ to $(2i,0)$ minus the total number of paths from $(0,0)$ to $(2i,-4)$ which is equal to

$$\begin{aligned}\binom{2i}{i}-\binom{2i}{i+2} &= \frac{(2i)!}{(i!)^2}-\frac{(2i)!}{(i+2)!(i-2)!} = \frac{(2i)!}{(i!)^2}\Big(1-\frac{i(i-1)}{(i+1)(i+2)}\Big)\\
&= \frac{(2i)!}{(i!)^2}\frac{(i+1)(i+2)-i(i-1)}{(i+1)(i+2)} = \frac{(2i)!}{(i!)^2}\frac{4k+2}{(i+1)(i+2)}\\
&= \frac{1}{i+2}\frac{1}{(i+1)!}\frac{(2i)!(4i+2)}{i!} = \frac{1}{i+2}\frac{1}{(i+1)!}\frac{(2i)!(4i+2)(i+1)}{(i+1)!}\\
&= \frac{1}{i+2}\frac{1}{(i+1)!}\frac{(2i)!(4i^2+6i+2)}{(i+1)!} = \frac{1}{i+2}\frac{1}{(i+1)!}\frac{(2i)!(2i+1)(2i+2)}{(i+1)!}\\
&= \frac{1}{i+2}\binom{2i+2}{i+1} = C_{i+1}.\end{aligned}$$

□

A.2 Lemma A.2

Lemma A.2.

a) The number $a_{n,k}$ of unit-sloped paths of length $2n$ with $s_{k'} \geq -1$ for all $k' = 0,1,2,\dots,2n$ which end at position $(2n,2k)$, i.e., at height $2k$, is given by $a_{n,k} = \frac{k+1}{n+1}\binom{2n+2}{n-k}$.

b) The number of item sequences σ of length $2n$ without condensations where $\mathrm{BF}[\sigma] = m$ for $m \in \{n, n+1, \dots, 2n\}$ is given by $a_{n,m-n}$.

Proof.

a) The proof is an immediate consequence of the bijection between Dyck paths of length $2n+2$ and path pairs of length n given in Deutsch and Shaprio (2001) and the included remark concerning the relaxation of the restriction of path pairs having to end in the same point. To this end, we first modify the bijection by omitting the appended u-step at the beginning and the appended d-step at the end of the Dyck path in order to facilitate recurring unit-sloped paths that are allowed to hit the level of -1. Moreover, since the number of path pairs of length n having endpoints $k\sqrt{2}$ apart is $a_{n,k}$, it follows from the bijection that the number of unit-sloped paths ending at height $2k$ is $a_{n,k}$.

b) A total number of m bins with $m \in \{n, n+1, \ldots, 2n\}$ is obtained when in an item sequence without condensations $m - n$ out of the n pairs of successive items are pairs of large items for which no matching small items can be found afterwards. Each such item sequence corresponds to a unit-sloped path of length $2n$ with $s_k \geq -1$ for $k = 0, 1, 2, \ldots, 2n$ ending at height $2(m-n)$ because each pair of large items contributes an amount of 2 to the total height achieved at the end of the path, and the result follows.

□

A.3 Proof of Lemma 3.4

a) Since for $2n$ items at least n and at most $2n$ bins are needed, $\mathrm{BF}(2n, m) = 0$ for $m < n$ and $m > 2n$. For the remaining m, we perform a reverse induction on m. The base case $m = 2n$ is valid because the only item sequence which needs $2n$ bins has $2n$ large items and it holds that $\sum_{k=2n-n}^{n} a_{n,k} = a_{n,n} = \frac{n+1}{n+1}\binom{2n+2}{0} = 1$. For the inductive step, let $\mathrm{BF}(2n, m) = \sum_{k=m-n}^{n} a_{n,k}$ be valid for some m with $2n \geq m > n$. We show that $\mathrm{BF}(2n, m-1) = \sum_{k=m-1-n}^{n} a_{n,k}$. Because of $m > n$, there must be a pair of large items starting at an odd position for which no matching small items follow in every item sequence with objective value m since otherwise these large items could be matched with small items and would fit into a bin contradicting $m > n$. Hence, we obtain for any item sequence with objective value m an item sequence with objective value $m - 1$ by replacing the first pair of large items starting at an odd position for which no matching small items follow with a pair of small items which in turn lead to a condensation. As a result, we have $\mathrm{BF}(2n, m-1) = \mathrm{BF}(2n, m) + |\Sigma_{add}|$ where Σ_{add} are the additional item sequences leading to objective value $m - 1$ which have *not* resulted from establishing a condensation in an item sequence with objective value m. These item sequences can be mapped to a unit-sloped path of length $2n$ not going below level -1 and ending at height $2(m-1-n)$. From Lemma A.2, we have that $|\Sigma_{add}| = a_{n,m-n-1}$. Together with the induction hypothesis we conclude that

$$\mathrm{BF}(2n, m-1) = \mathrm{BF}(2n, m) + |\Sigma_{add}| = \sum_{k=m-n}^{n} a_{n,k} + a_{n,m-n-1} = \sum_{k=m-n-1}^{n} a_{n,k}.$$

b) Since for $2n+1$ items at least $n+1$ and at most $2n+1$ bins are needed, $\mathrm{BF}(2n+1, m) = 0$ for $m < n+1$ and $m > 2n+1$. Notice that whenever $m > n+1$ for an item sequence of length $2n+1$, we have $m > n$ for the same item sequence where the last item is deleted. Thus, there must be a pair of large items beginning at an odd position in the truncated sequence from the same reasoning as in part a) of the proof. Objective value m with $n+1 < m \leq 2n+1$ for an item sequence of length $2n+1$ can be attained in two ways: First, BF needed $m-1$ bins after $2n$ items and the $2n+1$st item leads to the mth bin. Second, BF needed m bins after $2n$ items and the $2n+1$st item needs no new bin. In the first case, we have $\textsc{Bf}(2n, m-1)$ item sequences which must incur a new bin upon appending a large item; appending a small item would leave the objective value at m because there are at least two large items which could be matched with the small item. In the second case, $\textsc{Bf}(2n, m)$ item sequences will not incur a new bin upon appending a small item as this item can be matched with one of the large items; appending a large item would lead to objective value $m+1$ since after $2n$ items there can never be a bin with a small item only. We obtain

$$\begin{aligned}\textsc{Bf}(2n+1, m) &= \textsc{Bf}(2n, m-1) + \textsc{Bf}(2n, m)\\ &= \sum_{k=m-1-n}^{n} a_{n,k} + \sum_{k=m-n}^{n} a_{n,k} = 2\sum_{k=m-n}^{n} a_{n,k} + a_{n,m-1-n}.\end{aligned}$$

Objective value $n+1$ can be attained in three ways: First, $\textsc{Bf}$ needed n bins after $2n$ items and the $2n+1$st item is large leading to the $n+1$st bin. Second, $\textsc{Bf}$ needed n bins after $2n$ items and the $2n+1$st item is small leading to the $n+1$st bin. Third, $\textsc{Bf}$ needed $n+1$ bins after $2n$ items and the $2n+1$st item is small, but does not lead to a new bin. The first case is trivial. In the second case, we seek for the same item sequences because neither of them can exhibit a pair of large items starting in an odd position. In the third case, we seek for the item sequences of length $2n$ with objective value $n+1$ which have at least one pair of large items beginning at an odd position such that the appended small item does not incur a new bin. These item sequences are counted by $\textsc{Bf}(2n, n+1)$. We obtain

$$\begin{aligned}\textsc{Bf}(2n+1, n+1) &= \textsc{Bf}(2n, n) + \textsc{Bf}(2n, n) + \textsc{Bf}(2n, n+1)\\ &= \sum_{k=0}^{n} a_{n,k} + \sum_{k=0}^{n} a_{n,k} + \sum_{k=1}^{n} a_{n,k} = 3\sum_{k=0}^{n} a_{n,k} - a_{n,0}.\end{aligned}$$

A.4 Proof of Theorem 3.5

a) We show by two-dimensional induction on n and m that $\sum_{k=m-n}^{n} a_{n,k} = \binom{2n+1}{m+1}$. Recall that $n = 1, 2, \ldots$ and $m = n, n+1, \ldots, 2n$. The base case $n = 1$ and $m = n = 1$ is valid because it holds that $\sum_{k=0}^{1} a_{1,k} = a_{1,0} + a_{1,1} = 2 + 1 = 3 = \binom{2\cdot 1+1}{1+1} = \binom{3}{2} = 3$. In the first inductive step (on n with fixed $m = n$), we show that $\sum_{k=0}^{n} a_{n,k} = \binom{2n+1}{n+1}$ holds. From Shapiro (1976), we know that $\frac{1}{2}\binom{2(n+1)}{n+1} = \sum_{k=0}^{n} a_{n,k}$. The result follows from

$$\frac{1}{2}\binom{2(n+1)}{n+1} = \frac{1}{2}\frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{(2n+2)(2n+1)!}{2(n+1)n!(n+1)!} = \binom{2n+1}{n+1}.$$

In the second inductive step (on m with arbitrary n), we show that $\sum_{k=m-n}^{n} a_{n,k} = \binom{2n+1}{m+1}$

implies $\sum\limits_{k=m+1-n}^{n} a_{n,k} = \binom{2n+1}{m+2}$. This can be seen by the following calculations:

$$\sum_{k=m+1-n}^{n} a_{n,k} = \sum_{k=m-n}^{n} a_{n,k} - a_{n,m-n} = \binom{2n+1}{m+1} - \frac{m-n+1}{n+1}\binom{2n+2}{2n-m}$$

$$= \frac{(2n+1)!(n+1)(m+2) - (m-n+1)(2n+2)!}{(2n-m)!(m+2)!(n+1)}$$

$$= \frac{(2n+1)!}{(m+2)!(2n-m-1)!}\frac{(n+1)(m+2) - (m-n+1)(2n+2)}{(2n-m)(n+1)}$$

$$= \frac{(2n+1)!}{(m+2)!(2n-m-1)!} \cdot 1 = \binom{2n+1}{m+2}.$$

The result now immediately follows from the formula given in part a) of Lemma 3.4.

b) For $m = n+1$, we have

$$3\sum_{k=0}^{n} a_{n,k} - a_{n,0} \overset{a)}{=} 3\binom{2n+1}{n+1} - \frac{1}{n+1}\binom{2n+2}{n} = \frac{(2n+1)!}{(n+2)!(n+1)!}(3n^2+7n+4)$$

and

$$\binom{2n+3}{n+2} - \binom{2n+1}{n+1} = \frac{(2n+1)!}{(n+2)!(n+1)!}(3n^2+7n+4).$$

which together yields the desired relation for $m = n+1$.

For $n+1 < m \leq 2n+1$, we have

$$2\sum_{k=m-n}^{n} a_{n,k} + a_{n,m-n-1} \overset{a)}{=} 2\binom{2n+1}{m+1} + \frac{m-n}{n+1}\binom{2n+2}{2n-m+1}$$

$$= \frac{(2n+2)!}{(m+1)!(2n-m+1)!}\left(\frac{2(2n-m+1)}{2n+2} + \frac{m-n}{n+1}\right)$$

$$= \frac{(2n+2)!}{(m+1)!(2n-m+1)!}\left(\frac{2n+2}{2n+2}\right) = \binom{2n+2}{m+1}.$$

The result now immediately follows from the formula given in part b) of Lemma 3.4.

A.5 Lemma A.3

Lemma A.3. *For $n \le m \le 2n$ it holds that* $\sum_{i\ge 1} C_i\binom{2n-2i}{m-i} = \sum_{i\ge 1} C_{i-1}\binom{2n-2i+1}{m-i+1}$.

Proof. We show that $\sum_{i\ge 1} C_i\binom{2n-2i}{m-i} - \sum_{i\ge 1} C_{i-1}\binom{2n-2i+1}{m-i+1} = 0$. Notice that from the definition of the binomial coefficient, i ranges in $\{1, 2, \ldots, 2n-m\}$ in both terms. Hence,

$$
\begin{aligned}
&\sum_{i\ge 1} C_i\binom{2n-2i}{m-i} - \sum_{i\ge 1} C_{i-1}\binom{2n-2i+1}{m-i+1} \\
= \quad & C_1\binom{2n-2}{m-1} + C_2\binom{2n-4}{m-2} + C_3\binom{2n-6}{m-3} + \ldots + C_{2n-m}\binom{2m-n}{2m-n} \\
& - \left(C_0\binom{2n-1}{m} + C_1\binom{2n-3}{m-1} + C_2\binom{2n-5}{m-2} + \ldots + C_{2n-m-1}\binom{2m-n+1}{2m-n+1}\right) \\
= \quad & C_1\binom{2n-3}{m-2} + C_2\binom{2n-5}{m-3} + \ldots + C_{2n-m-1}\binom{2m-n+1}{2m-2n} + C_{2n-m} - \binom{2n-1}{m} \\
= \quad & \sum_{i=1}^{2n-m-1} C_i\binom{2n-2i-1}{m-i-1} + C_{2n-m} - \binom{2n-1}{m} \\
= \quad & \sum_{i=0}^{2n-m-1} C_i\binom{2n-2i-1}{m-i-1} - \binom{2n-1}{m-1} + C_{2n-m} - \binom{2n-1}{m} \\
= \quad & \sum_{i=0}^{2n-m} C_i\binom{2n-2i-1}{m-i-1} - C_{2n-m}\binom{2m-2n-1}{2m-2n-1} - \binom{2n-1}{m-1} + C_{2n-m} - \binom{2n-1}{m} \\
= \quad & \sum_{i\ge 0} C_i\binom{2n-2i-1}{m-i-1} - C_{2n-m} - \binom{2n-1}{m-1} + C_{2n-m} - \binom{2n-1}{m} \\
= \quad & \binom{2n}{m} - \binom{2n-1}{m-1} - \binom{2n-1}{m} = \binom{2n}{m} - \binom{2n}{m} = 0.
\end{aligned}
$$

$\square$

A.6 Proof of Corollary 3.7

From Theorem 3.1, we know that $\mathrm{BF}_2[\sigma] \in \{m, m-1\}$ whenever $\mathrm{BF}[\sigma] = m$; from the previous Theorem 3.6, we know that $|\{\sigma \,|\, |\sigma| = n, \mathrm{BF}_2[\sigma] = m-1, \mathrm{BF}[\sigma] = m\}| = \binom{n}{m+1}$ for $m = \lceil \frac{n}{2} \rceil + 1, \ldots, n-1$. Hence, from Theorem 3.5 it immediately follows for these m that $|\{\sigma \,|\, |\sigma| = n, \mathrm{BF}_2[\sigma] = m\}| = \binom{n+1}{m+1} - \binom{n}{m+1} = \binom{n}{m}$. Clearly, $|\{\sigma \,|\, |\sigma| = n, \mathrm{BF}[\sigma] = n\}| = |\{\sigma \,|\, |\sigma| = n, \mathrm{BF}_2[\sigma] = n\}| = 1$.

A.7 Proof of Corollary 3.10

a) For $v < n$ and $v \geq 2n-1$, $F_{\mathrm{BF}}(v) = F_{\mathrm{BF}_2}(v)$. For $n \leq v < 2n-1$

$$F_{\mathrm{BF}_2}(v) - F_{\mathrm{BF}}(v) = \left(\sum_{m=n}^{\lfloor v \rfloor} \left(\binom{2n+1}{m+2} - 2\binom{2n}{m+1} \right) + \binom{2n}{n+1} \right) \cdot (2^{-2n})$$

$$= \left(\sum_{m=n}^{\lfloor v \rfloor} \left(\binom{2n}{m+2} - \binom{2n}{m+1} \right) + \binom{2n}{n+1} \right) \cdot (2^{-2n}) = \binom{2n}{\lfloor v \rfloor + 2} \cdot (2^{-2n}) > 0.$$

The second part follows immediately from Pascal's triangle as a result of $\binom{2n}{v+2} > \binom{2n}{v+3}$ for $v = n, n+1, \ldots, 2n-3$.

b) Using the formula of Stirling ($n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$, KÃ¶nigsberger (2001)), we get for $n \to \infty$ that

$$F_{\mathrm{BF}_2}(n) - F_{\mathrm{BF}}(n) = \binom{2n}{n+2} \cdot 2^{-2n} \approx \frac{\sqrt{4\pi n}(\frac{2n}{e})^{2n}}{\sqrt{2\pi(n+2)}(\frac{n+2}{e})^{n+2}\sqrt{2\pi(n-2)}(\frac{n-2}{e})^{n-2}} \cdot 2^{-2n}$$

$$= \frac{\sqrt{n}2^{2n}n^{2n}}{\sqrt{\pi(n^2-4)}(n+2)^{n+2}(n-2)^{n-2}} \cdot 2^{-2n} \in \Theta(\frac{1}{\sqrt{n}}).$$

In addition, $\binom{2n}{v} > \binom{2n}{v+1}$ for $v \in \mathbb{N}$ with $v \geq n$, i.e., $F_{\mathrm{BF}_2}(v) - F_{\mathrm{BF}}(v)$ monotonously decreases for $v \geq n$. Since also $F_{\mathrm{BF}_2}(v) = F_{\mathrm{BF}}(v)$ for $v < n$ and $v \geq 2n-1$, the result follows.